Stellarator Expansion at Finite Aspect Ratio

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To Professor Arnulf Schlüter on his 60th Birthday

A previous stellarator expansion valid for finite aspect ratio and small rotational transform ι and based on the axisymmetric toroidal vacuum field is generalized to stellarator equilibria with arbitrary closed line vacuum fields as zeroth order field. The equilibrium beta value behaves as $\beta \sim \iota^2$. The condition for the magnetic surfaces to be unaffected by β within this ordering is formulated.

I. Introduction

Various stellarator expansions have hitherto been obtained for large aspect ratio by many authors (see, e.g. [1–3]). Recently, a low-β stellarator expansion at finite aspect ratio was devised [4], which was based on the axisymmetric toroidal vacuum field as zeroth order field. Here, we generalize this type of expansion to arbitrary closed line vacuum fields as zeroth order field. A prerequisite for the equilibrium expansion is the asymptotic expansion of magnetic surfaces at small values of the rotational transform [5, 6] which has been obtained in an explicit form [7] lending itself to application. Stellarator expansions at finite aspect ratio should be particularly suitable for the investigation of separatrix formation and its relation to the β-value [8, 9]. Arbitrary closed line vacuum fields as zeroth order are necessary for an adequate description of toroidal equilibria with significantly reduced parallel current density [10, 11].

II. Equilibrium Expansion for Small Rotational Transform

The MHD equilibrium equations are written in the form

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{1}$$

$$\boldsymbol{B} \cdot \nabla F = 0\,, \tag{2}$$

$$\boldsymbol{B} \cdot \nabla j = (\mathrm{d}p/\mathrm{d}F)\,\nabla \cdot (\nabla F \times \boldsymbol{B}/B^2)\,, \tag{3}$$

$$\boldsymbol{j} = \nabla \times \boldsymbol{B} \tag{4}$$

where F describes the magnetic surfaces, $j = \boldsymbol{j} \cdot \boldsymbol{B}/B^2$ is related to the parallel current density; (3) guarantees that $\boldsymbol{j}$ is divergencefree, so that (4) is integrable.

The following ordering is employed

$$\begin{aligned} \boldsymbol{B} &= \boldsymbol{B}_0 + \varepsilon \boldsymbol{B}_1 + \varepsilon^2 \boldsymbol{B}_2 + \cdots, \\ F &= F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots, \\ \iota &\sim 0(\varepsilon^2)\,, \\ \beta &\sim p/B^2 \sim 0(\varepsilon^4)\,, \end{aligned} \tag{5}$$

where ι is the rotational transform and β characterizes the ratio of thermal to magnetic energy.

$\boldsymbol{B}_0$ is taken as a vacuum field with toroidally closed lines, so that it may be represented as

$$\boldsymbol{B}_0 = \nabla \varphi = \nabla \psi \times \nabla \chi \tag{6}$$

with single-valued functions ψ and χ. In the following, ψ, χ, and φ are used as independent variables (coordinates). The following relations hold

$$\begin{aligned} D &= (\nabla \psi \times \nabla \chi) \cdot \nabla \varphi = B_0{}^2 = 1/\sqrt{g}\,, \\ B_0 &= D\,\boldsymbol{r}_{,\varphi}\,, \quad \boldsymbol{B}_0 \cdot \nabla = D\,\partial_\varphi\,, \\ g^{\psi\varphi} &= g^{\chi\varphi} = g_{\psi\varphi} = g_{\chi\varphi} = 0\,. \end{aligned} \tag{7}$$

The first and higher order fields are represented as in [7] by

$$\boldsymbol{B}_\nu = \nabla u_\nu \times \nabla \psi - \nabla v_\nu \times \nabla \chi\,, \quad \nu \geqq 1\,. \tag{8}$$

Since we are interested in a stellarator expansion B_1 is the curlfree leading order stellarator field with single-valued u_1 and v_1 in accordance with (5). With (6) and (8), (1) is automatically satisfied, i.e. $\boldsymbol{B}$ divergencefree. In ψ, χ, φ coordinates

$$\begin{aligned} \boldsymbol{B}_\nu \cdot \nabla = D[&v_{\nu,\varphi}\,\partial_\psi + u_{\nu,\varphi}\,\partial_\chi \\ &- (u_{\nu,\chi} + v_{\nu,\psi})\,\partial_\varphi] \end{aligned} \tag{9}$$

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is obtained, so that (2) reads

$$\partial_\varphi F_0 = 0\,,$$
$$\partial_\varphi F_\mu + \sum_{\nu=1}^{\mu} [v_{\nu,\varphi}\,\partial_\psi + u_{\nu,\varphi}\,\partial_\chi - (u_{\nu,\chi} + v_{\nu,\psi})\,\partial_\varphi] F_{\mu-\nu} = 0\,. \quad (10)$$

The results

$$F_0 = F_0(\psi,\chi)\,,$$
$$F_1 = f_1 + g_1\,, \quad f_1 = -\,v_1 F_{0,\psi} - u_1 F_{0,\chi}\,,$$
$$g_1 = g_1(\psi,\chi) \quad (11)$$

are easily obtained [7].

Since $\boldsymbol{B}_0$ and $\boldsymbol{B}_1$ are curlfree, the Eqs. (5) are supplemented by

$$\boldsymbol{j} = \varepsilon^2 \boldsymbol{j}_2 + \varepsilon^3 \boldsymbol{j}_3 + \cdots\,,$$
$$j = \varepsilon^2 j_2 + \varepsilon^3 j_3 + \cdots\,, \quad (12)$$

so that the leading order of (3) reads

$$\partial_\varphi j_2 = 0$$

with the result

$$j_2 = j_2(\psi,\chi)\,. \quad (13)$$

The third order of (3) yields

$$\partial_\varphi j_3 + u_{1,\varphi}\, j_{2,\chi} + v_{1,\varphi}\, j_{2,\psi} = 0\,,$$

so that

$$j_3 = -\,u_1 j_{2,\chi} - v_1 j_{2,\psi} + \bar{j}_3(\psi,\chi)\,. \quad (14)$$

Because of the ordering of β, the fourth order of (3) is obtained as ($p' = \mathrm{d}p/\mathrm{d}F_0$)

$$D\,\partial_\varphi j_4 + \boldsymbol{B}_1 \cdot \nabla j_3 + \boldsymbol{B}_2 \cdot \nabla j_2 = D p' \mathbf{D} \frac{1}{D}\,,$$

where $\mathbf{D} = F_{0,\chi}\,\partial_\psi - F_{0,\psi}\,\partial_\chi$ differentiates parallel to $F_0 = \text{const}$, i.e. poloidally.

Its solubility condition $[\langle\cdots\rangle = \int \mathrm{d}\varphi/D(\cdots)]$

$$\langle \boldsymbol{B}_1 \cdot \nabla j_3 + \boldsymbol{B}_2 \cdot \nabla j_2 \rangle = p' \langle D \mathbf{D}\, 1/D \rangle = p' \mathbf{D}\, Q = \mathbf{D}\, p' Q\,, \quad (15)$$

where $Q = \int \mathrm{d}\varphi/D = \int \mathrm{d}l/B_0$ and the integral is performed along the zeroth order field lines, is the leading order equilibrium equation.

The evaluation of (15) is performed in two steps. First,

$$\langle \boldsymbol{B}_1 \cdot \nabla j_3 \rangle = \langle D (j_{2,\psi}\,\partial_\chi - j_{2,\chi}\,\partial_\psi)(u_1 v_{1,\varphi}) \rangle$$

is obtained with the help of (14). Second, the flux of the second order field B_2 through two lines of the zeroth order field is introduced

$$U = \iint \boldsymbol{B}_2 \cdot \mathrm{d}^2 f = \int_{\psi=\text{const}} D^{-1} B_2{}^\psi\, \mathrm{d}\varphi\, \mathrm{d}\chi - \int_{\chi=\text{const}} D^{-1} B_2{}^\chi\, \mathrm{d}\varphi\, \mathrm{d}\psi \quad (16)$$

to compute

$$-\,j_{2,\chi}\, U_{,\psi} + j_{2,\psi}\, U_{,\chi} = \langle \boldsymbol{B}_2 \cdot \nabla j_2 \rangle\,.$$

Now the result [7]

$$-\,U - \int u_1 v_{1,\varphi}\, \mathrm{d}\varphi = H(F_0) \quad (17)$$

is employed, i.e. that this combination only depends on F_0 (for the interpretation of H as second order poloidal flux, see Appendix). Because of

$$(j_{2,\psi}\,\partial_\chi - j_{2,\chi}\,\partial_\psi) H = H' \mathbf{D}\, j_2\,,$$

(15) may be integrated to give

$$j_2 = \bar{j}_2(H) - Q\, \mathrm{d}p/\mathrm{d}H\,. \quad (18)$$

The structure of this equilibrium problem may be elucidated by the following iteration scheme

$$\overset{(n)}{F_0}(\psi,\chi) \Rightarrow \overset{(n)}{j_2}(\psi,\chi) \Rightarrow \overset{(n)}{\boldsymbol{B}_2} \Rightarrow \overset{(n)}{U} \Rightarrow \overset{(n+1)}{F_0}\,, \quad (19)$$

where the first step is accomplished by (18) [for given $\mathrm{d}p/\mathrm{d}H$ and $\bar{j}_2(H)$]; the second step involves solving (4), e.g. by obtaining the vector potential of $\boldsymbol{B}_2$ via Poisson's integral with $\boldsymbol{j}_2$ as kernel; the third and the fourth step are explicitly given by (16) and (17). While the second step in general is a three-dimensional problem, it becomes two-dimensional in special cases, e.g. if the zeroth order field is the axisymmetric toroidal vacuum field, see [4].

With regard to stellarators, the case of vanishing net toroidal current J through each magnetic surface is of special interest. In leading order

$$J = \varepsilon^2 J_2 = \varepsilon \int_{F_0} j_2\, \mathrm{d}\psi\, \mathrm{d}\chi\,,$$

so that this current can be made zero by appropriate choice of $\bar{j}_2(H)$ in (18). In an iterative procedure following (19) this would be part of the first step.

III. Discussion

In the expansion set forth above the equilibrium beta value behaves as

$$\beta \sim 0(\varepsilon^4) \sim \iota^2\,. \quad (19)$$

An interesting special case occurs if

$$Q = Q(F_{0,\text{vac}})\,, \quad (20)$$

where $F_{0,\text{vac}}$ describes the zeroth order magnetic surfaces as obtained from the vacuum stellarator fields $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$. Equation (18) shows that j_2, i.e. the leading order parallel current density, then vanishes identically if J_2 does. Thus, the zeroth order magnetic surfaces are unaffected by β [within the ordering (19)] if (20) holds. In particular, there then is no Shafranov shift in accordance with the results obtained with 3D codes [10, 11] for vacuum fields obeying a relation similar to (20).

Another interesting consequence arises with respect to MHD stability if (20) is satisfied. Normally, the stability behaviour of a stellarator is intricate even with the orderings used, because the magnetic well and other terms in the stability criteria (see, e.g. [12]) depend on β, since F_0 depends on β. If (20) holds, the stability behaviour, for β given by the ordering (19), is completely determined by the vacuum magnetic field. In particular, stability holds [12] if there exists a vacuum magnetic well ($\dot{\Phi}\ddot{\Phi} > 0$, Φ longitudinal flux, $\dot{} = \mathrm{d}/\mathrm{d}V$, V volume of zeroth order surfaces F_0).

Finally, one may ask whether or not (20) allows to order β larger than has been done here. An obvious conjecture is $\beta \sim 0(1)$, which, however, would not allow to start the expansion from a vacuum field, see (6), but only from finite-β equilibria with zero transform. In such equilibria the surfaces $Q =$ const are determined and no freedom would be left to make them coincide with the zeroth order magnetic surfaces, see (17). So, the possibility of an expansion with $\beta \sim 0(1)$ seems unlikely. Future work investigating this problem more closely will use an ordering in which β and ι are small but independent [in contrast to (5)] if (20) holds.

Appendix

Here, we prove that (17), viz.

$$U + \int v_1 u_{1,\varphi} \mathrm{d}\varphi \tag{21}$$

is the leading order poloidal flux. First, the equations for the perturbed field lines

$$\bar{\psi} = \psi + \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \cdots = \text{const},$$
$$\bar{\chi} = \chi + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \cdots = \text{const}$$

are obtained from

$$\boldsymbol{B} \cdot \nabla \bar{\psi} = 0,$$
$$\boldsymbol{B} \cdot \nabla \bar{\chi} = 0,$$

via the representation of $\boldsymbol{B}_\nu$, $\nu \geqq 1$, see (8). We obtain

$$\psi_1 = -v_1,$$
$$\chi_1 = -u_1,$$
$$\psi_{2,\varphi} = -v_{2,\varphi} + u_{1,\varphi} v_{1,\chi} - u_{1,\chi} v_{1,\varphi} = -D^{-1} B_2{}^{\psi} + u_{1,\varphi} v_{1,\chi} - u_{1,\chi} v_{1,\varphi},$$
$$\chi_{2,\varphi} = -u_{2,\varphi} + v_{1,\varphi} u_{1,\psi} - v_{1,\psi} u_{1,\varphi} = -D^{-1} B_2{}^{\chi} + v_{1,\varphi} u_{1,\psi} - v_{1,\psi} u_{1,\varphi},$$

so that the displacement of the field lines after one toroidal turn $\varphi + \Delta\varphi$ is given by [see (16)]

$$\psi_2(\varphi + \Delta\varphi) = -\partial_\chi (U + \textstyle\int u_1 v_{1,\varphi} \mathrm{d}\varphi),$$
$$\chi_2(\varphi + \Delta\varphi) = \partial_\psi (U + \textstyle\int u_1 v_{1,\varphi} \mathrm{d}\varphi).$$

Second, we take ψ as label for the zeroth order magnetic surfaces and χ as poloidal variable. Then $\psi_2 \equiv 0$ and the poloidal flux can be obtained as the flux of the zeroth order field through the second order band (at $\varphi =$ const) which is given by the starting line of field lines at $\chi =$ const and the image line $\chi + \varepsilon^2 \chi_2$ obtained after one toroidal turn:

$$\int \boldsymbol{B}_0 (\boldsymbol{r}_{,\psi} \times \boldsymbol{r}_{,\chi}) \mathrm{d}\psi \mathrm{d}\chi = \int \mathrm{d}\chi \mathrm{d}\psi = \varepsilon^2 \int \chi_2 \mathrm{d}\psi = \varepsilon^2 (U + \textstyle\int u_1 v_{1,\varphi} \mathrm{d}\varphi).$$

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